APFEL
A PDF Evolution Library with QED corrections
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Stefano Carrazza

University & INFN Milan

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APFEL

in collaboration with Valerio Bertone and Juan Rojo

Outline

1. Motivation
   - Why APFEL?
   - APFEL features
2. DGLAP evolution with QED corrections
   - Solving the QED evolution equations
   - QCD$\otimes$QED combined evolution in the VFN scheme
3. Validation and Benchmarks
   - QCD evolution
   - QED evolution
   - APFEL timing
4. Conclusion

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Problem: No public code for DGLAP with QED corrections.

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Pure QCD:

- HOPPET
- QCDNUM
- PEGASUS
- ...

Combined QCD+QED:

- partonevolution:
  - fixed-flavor-number scheme (FFNS)
  - NLO in QCD and LO in QED
  - uses hardcoded toy PDF (N-space appr.)
- MRST2004QED:
  - no public code
  - not possible to change initial conditions.

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Solution: APFEL a new public code for combined DGLAP evolution

APFEL is *"A PDF Evolution Library" with QED corrections*

APFEL features

APFEL 1.0.0 features:

- QCD DGLAP, QED DGLAP
- innovative methodology for the QCD $\otimes$ QED solution
  - possibility to explore $\mathcal{O}(\alpha\alpha_s)$ terms
- variable-flavor-number scheme (VFNS) and FFNS
- x-space solution
- pole and $\overline{\text{MS}}$ schemes for the heavy quark masses
- interface to LHAPDF (input/output)
- code interfaces in Fortran, C/C++ and Python

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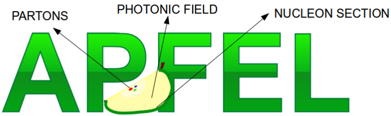
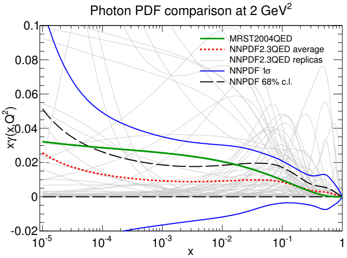
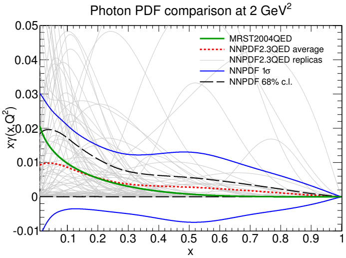


APFEL and NNPDF2.3QED

- APFEL and NNPDF:
  - independent cross-check implementation of the FastKernel code
- QED evolution in new NNPDF2.3QED set:





- Photon PDF extracted from DIS data and LHC data.
  - LO in QED and up to NNLO in QCD ($\alpha_s = 0.117, 0.118, 0.119$)
  - public PDF sets available from LHAPDF$\geq$ 5.9.0.

Outline

Solving the QED evolution equations

- The approach followed by `partonevolution` and `MRST2004QED` consists in solving explicitly the equation:

$$\mu^2 \frac{\partial}{\partial \mu^2} \mathbf{q}(x,\mu) = \left[\mathbf{P}^{\text{QED}}(x, \alpha(\mu)) + \mathbf{P}^{\text{QCD}}(x, \alpha_s(\mu)) \right] \otimes \mathbf{q}(x,\mu)$$

  where $\mathbf{P}^{\text{QCD}}$ and $\mathbf{P}^{\text{QED}}$ are respectively the QCD and QED matrices of splitting functions, and $\mathbf{q}(x,\mu)$ is a vector containing all the PDFs.

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where $\mathbf{P}^{\text{QCD}}$ and $\mathbf{P}^{\text{QED}}$ are respectively the QCD and QED matrices of splitting functions, and $\mathbf{q}(x,\mu)$ is a vector containing all the PDFs.

`APFEL` solves the coupled QCD⊗QED DGLAP with

- QED corrections up to $\mathcal{O}(\alpha)$, neglecting $\mathcal{O}(\alpha\alpha_s)$
- using a factorized approach

Solving the QED evolution equations

- Lets denote:
  - $q_i \equiv q_i(x, \mu, \nu)$ the i-th parton distribution function
  - μ and ν respectively the factorization scales of QCD and QED

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\nu^2 \frac{\partial}{\partial \nu^2} \mathbf{q}(x, \mu, \nu) &= \mathbf{P}^{\text{QED}}(x, \alpha(\nu)) \otimes \mathbf{q}(x, \mu, \nu)
\end{aligned}
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Solving the QED evolution equations

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\end{aligned}$$

- The decoupled DGLAP equations from $\mu_0 \to \mu_1, \nu_0 \to \nu_1$ are solved by

$$\begin{aligned}
\mathbf{q}(x,\mu_1,\nu) &= \mathbf{\Gamma}^{\text{QCD}}(x|\mu_1,\mu_0) \otimes \mathbf{q}(x,\mu_0,\nu) \\
\mathbf{q}(x,\mu,\nu_1) &= \mathbf{\Gamma}^{\text{QED}}(x|\nu_1,\nu_0) \otimes \mathbf{q}(x,\mu,\nu_0)
\end{aligned}$$

Combining the QCD and QED evolution operators

- When combining QCD and QED evolution operators we have

$$\left[\mathbf{\Gamma}^{\mathrm{QCD}},\mathbf{\Gamma}^{\mathrm{QED}}\right]\neq 0,$$

this implies that

$$\mathbf{\Gamma}^{\mathrm{QECD}}(\mu,\mu_0;\nu,\nu_0)\otimes\mathbf{q}(\mu_0,\nu_0)\neq\mathbf{\Gamma}^{\mathrm{QCED}}(\mu,\mu_0;\nu,\nu_0)\otimes\mathbf{q}(\mu_0,\nu_0),$$

where

$$\begin{aligned}\mathbf{\Gamma}^{\mathrm{QCED}}(\mu,\mu_0;\nu,\nu_0) &\equiv \mathbf{\Gamma}^{\mathrm{QED}}(\nu,\nu_0)\otimes\mathbf{\Gamma}^{\mathrm{QCD}}(\mu,\mu_0),\\ \mathbf{\Gamma}^{\mathrm{QECD}}(\mu,\mu_0;\nu,\nu_0) &\equiv \mathbf{\Gamma}^{\mathrm{QCD}}(\mu,\mu_0)\otimes\mathbf{\Gamma}^{\mathrm{QED}}(\nu,\nu_0).\end{aligned}$$

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where

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- Using the analytical solution of the QCD and QED DGLAP equations in Mellin space and the Baker-Campbell-Hausdorff formula, we show that:

$$\left[\mathbf{\Gamma}^{\text{QCD}}, \mathbf{\Gamma}^{\text{QED}}\right] = \mathcal{O}(\alpha\alpha_s)$$

in particular, we have

$$\begin{aligned} \mathbf{\Gamma}^{\text{QCED}} &= \mathbf{1} + \alpha \boldsymbol{A} + \alpha_s \boldsymbol{B} + \alpha\alpha_s \boldsymbol{C} + \mathcal{O}(\alpha^2), \\ \mathbf{\Gamma}^{\text{QECD}} &= \mathbf{1} + \alpha \boldsymbol{A} + \alpha_s \boldsymbol{B} - \alpha\alpha_s \boldsymbol{C} + \mathcal{O}(\alpha^2). \end{aligned}$$

- These expansions suggest a third possibility:

$$\mathbf{\Gamma}^{\text{QavD}} \equiv \left(\mathbf{\Gamma}^{\text{QCED}} + \mathbf{\Gamma}^{\text{QECD}}\right)/2$$

QCD$\otimes$QED combined evolution in the VFN scheme

- Lets consider the crossing of the charm mass threshold m_c ($i.e.$ $\mu_0, \nu_0 < m_c < \mu, \nu$)

$$
\begin{aligned}
\mathbf{q}(\mu, \nu_0) &= \mathbf{\Gamma}^{\text{QCD},(4)}(\mu, m_c) \otimes \mathbf{\Gamma}^{\text{QCD},(3)}(m_c, \mu_0) \otimes \mathbf{q}(\mu_0, \nu_0), \\
\mathbf{q}(\mu_0, \nu) &= \mathbf{\Gamma}^{\text{QED},(4)}(\nu, m_c) \otimes \mathbf{\Gamma}^{\text{QED},(3)}(m_c, \nu_0) \otimes \mathbf{q}(\mu_0, \nu_0).
\end{aligned}
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\end{aligned}
$$

- We have two options for the combined QCD ⊗ QED evolution:

$$
\begin{aligned}
\mathbf{q}(\mu, \nu) &= \left\{ \left[\mathbf{\Gamma}^{\text{QED},(4)}(\nu, m_c) \otimes \mathbf{\Gamma}^{\text{QCD},(4)}(\mu, m_c) \right] \otimes \right. \\
& \left. \left[\mathbf{\Gamma}^{\text{QED},(3)}(m_c, \nu_0) \otimes \mathbf{\Gamma}^{\text{QCD},(3)}(m_c, \mu_0) \right] \right\} \otimes \mathbf{q}(\mu_0, \nu_0) \\
&\equiv \mathbf{\Gamma}^{\text{QCEDP}}(\mu, \mu_0; \nu, \nu_0) \otimes \mathbf{q}(\mu_0, \nu_0) \qquad \text{(3x parallel)}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{q}(\mu, \nu) &= \left\{ \left[\mathbf{\Gamma}^{\text{QED},(4)}(\nu, m_c) \otimes \mathbf{\Gamma}^{\text{QED},(3)}(m_c, \nu_0) \right] \otimes \right. \\
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&\equiv \mathbf{\Gamma}^{\text{QCEDS}}(\mu, \mu_0; \nu, \nu_0) \otimes \mathbf{q}(\mu_0, \nu_0) \qquad \text{(3x series)}
\end{aligned}
$$

- There is no need to keep the QCD and the QED factorization scales different and thus in `APFEL` they are always taken to be equal, $\mu = \nu = Q$.

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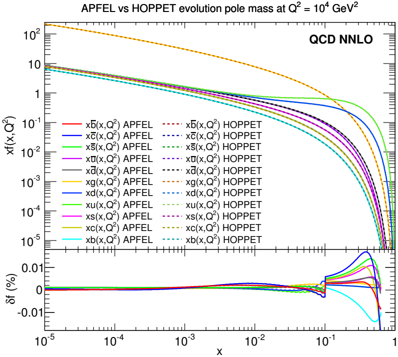
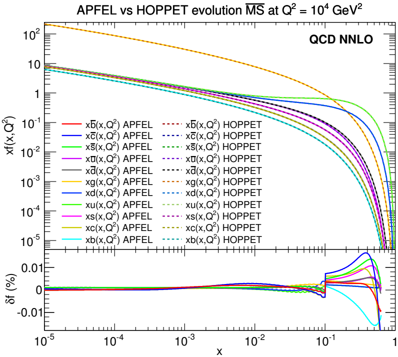
   - APFEL timing
4. Conclusion

- **Example:** APFEL vs HOPPET
  - initial condition: Les Houches toy PDF
  - VFNS, NNLO in QCD, from $Q_0^2 = 2$ GeV^2 up to $Q^2 = 10^4$ GeV^2
  - excellent agreement for the whole range in x



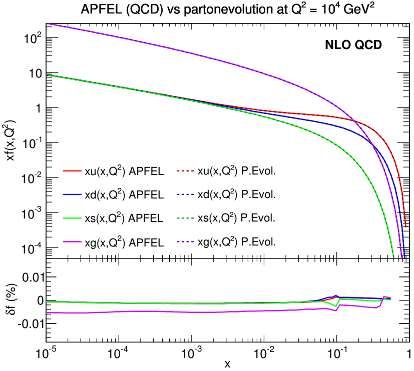
(a) Pole mass scheme



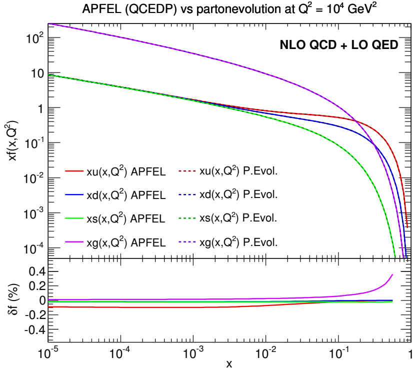
(b) $\overline{\text{MS}}$ scheme

APFEL vs partonevolution

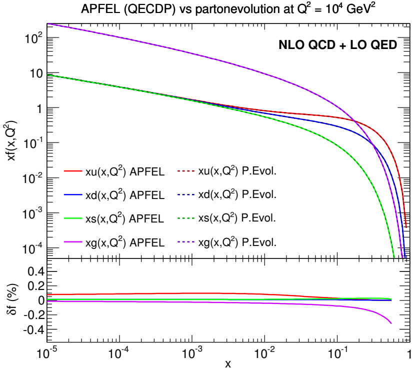
- **Example:** APFEL vs partonevolution 1.1.3
  - initial condition: partonevolution 1.1.3 hard-coded toy PDF
  - FFNS, NLO in QCD, from $Q_0^2 = 4$ GeV^2 up to $Q^2 = 10^4$ GeV^2
  - good agreement for the whole range in x



- **Example:** `APFEL` vs `partonevolution 1.1.3`
  - initial condition: `partonevolution 1.1.3` hard-coded toy PDF
  - FFNS, NLO in QCD + LO QED, $Q_0^2 = 4$ GeV^2, $Q^2 = 10^4$ GeV^2

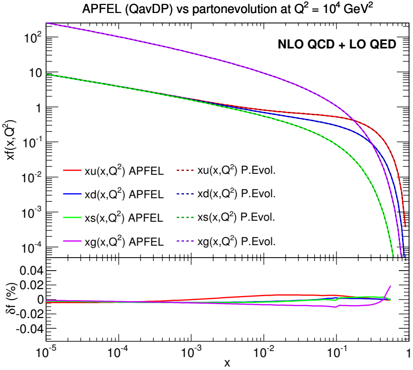


(c) `APFEL` QCEDP

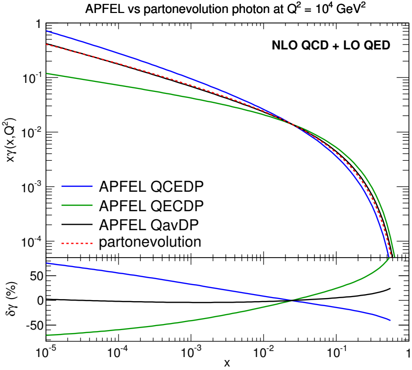


(d) `APFEL` QECDP

- Photon PDF at initial scale: $\gamma(x, Q_0^2) = 0$.
  - APFEL best agreement with partonevolution with QavDP
  - Spread can be used as an estimator of theoretical uncertainties

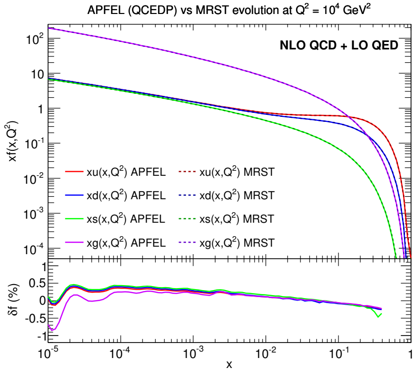


(e) APFEL QavDP

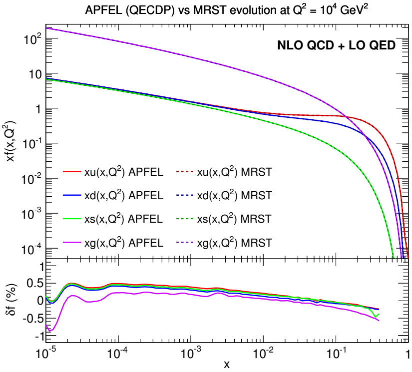


(f) Photon PDF solutions

- **Example:** APFEL vs MRST2004QED
  - initial condition: MRST2004QED LHgrid @ $Q^2 = 2$ GeV^2.
  - VFNS, NLO in QCD + LO QED, $Q_0^2 = 2$ GeV^2, $Q^2 = 10^4$ GeV^2

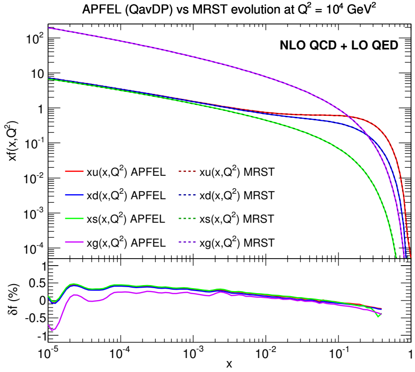


(g) APFEL QCEDP

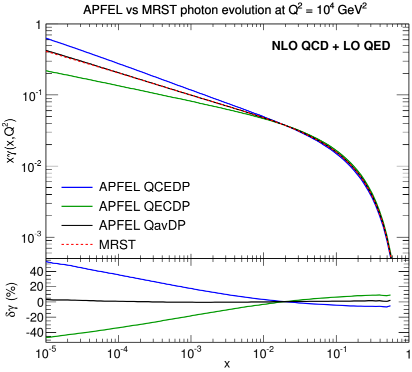


(h) APFEL QECDP

- Photon PDF at initial scale: MRST2004QED.
  - APFEL best agreement with MRST2004QED with QavDP
  - Spread can be used as an estimator of theoretical uncertainties



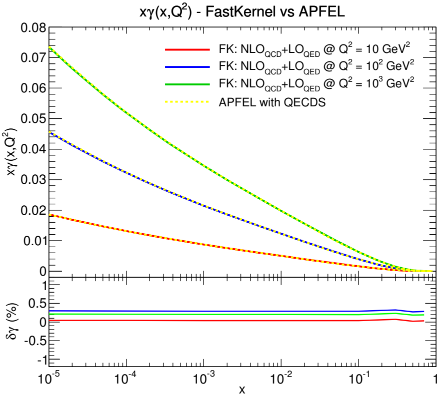
(i) APFEL QavDP



(j) Photon PDF solutions

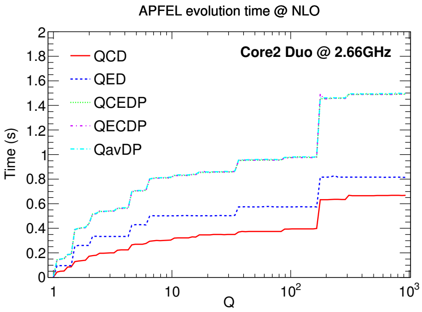
APFEL vs FastKernel

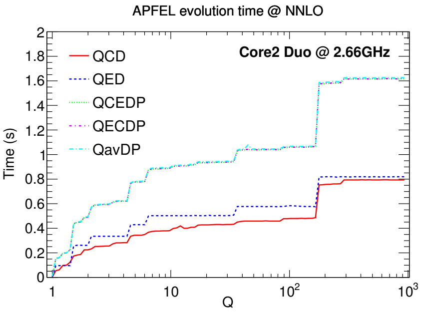
- APFEL vs FastKernel codes
  - $\gamma(x, Q_0^2 = 2\,\text{GeV}^2) = 0$
  - FastKernel iterated solution
  - flat constant difference due to LHAPDF interpolation
  - excellent agreement



APFEL timing

- APFEL is ~100x faster than partonevolution
- Smooth increase of computation time in function of Q
- Same computation time for QavD, QCED and QECD solutions





Outline

Outlook

- **Conclusion**
  - Public library for combined QCD+QED evolution
    - up to NNLO in QCD and LO in QED in VFNS
  - Modern approach for PDF manipulation
    - interface to LHAPDF
  - Good agreement with previous codes, possibility to estimate theoretical uncertainties due to subleading terms.
- **Outlook**
  - DIS scheme
  - DGLAP for polarized partons
  - implementation of $\alpha\alpha_S$ splitting functions.
- **Code/Manual:**

`http://apfel.hepforge.org/`

Solving the QED evolution equations

- At leading order, the QED DGLAP (dropping for simplicity μ) is:

$$
\begin{aligned}
\nu^2 \frac{\partial}{\partial \nu^2}\gamma(x,\nu) &= \frac{\alpha(\nu)}{4\pi}\left[\left(\sum_i N_c e_i^2\right) P_{\gamma\gamma}^{(0)}(x)\otimes\gamma(x,\nu)\right. \\
&\qquad \left. + \sum_i e_i^2 P_{\gamma q}^{(0)}(x)\otimes(q_i+\bar{q}_i)(x,\nu)\right] \\
\nu^2 \frac{\partial}{\partial \nu^2} q_i(x,\nu) &= \frac{\alpha(\nu)}{4\pi}\left[N_c e_i^2 P_{q\gamma}^{(0)}(x)\otimes\gamma(x,\nu) + e_i^2 P_{qq}^{(0)}(x)\otimes q_i(x,\nu)\right] \\
\nu^2 \frac{\partial}{\partial \nu^2} \bar{q}_i(x,\nu) &= \frac{\alpha(\nu)}{4\pi}\left[N_c e_i^2 P_{q\gamma}^{(0)}(x)\otimes\gamma(x,\nu) + e_i^2 P_{qq}^{(0)}(x)\otimes \bar{q}_i(x,\nu)\right]
\end{aligned}
$$

where $\gamma(x,\nu)$, $q_i(x,\nu)$ and $\bar{q}_i(x,\nu)$ are the PDFs of the photon, the i-th quark and antiquark, e_i the quark electric charge, $N_c = 3$ the color factor.

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&\qquad \left. + \sum_i e_i^2 P^{(0)}_{\gamma q}(x)\otimes(q_i+\bar{q}_i)(x,\nu)\right] \\
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- The QED LO splitting functions are

$$P^{(0)}_{q\gamma}(x) = 2\left[x^2 + (1-x)^2\right], \qquad P^{(0)}_{\gamma q}(x) = 2\left[\frac{1+(1-x)^2}{x}\right],$$

$$P^{(0)}_{\gamma\gamma}(x) = -\frac{4}{3}\delta(1-x), \qquad P^{(0)}_{qq}(x) = 2\frac{1+x^2}{(1-x)_+} + 3\delta(1-x)$$

Solving the QED evolution equations

- In APFEL we adopt the following PDF basis for the QED DGLAP
  - Singlet: $\mathbf{q}^{\text{SG}} = \begin{pmatrix} \gamma \\ \Sigma \equiv u^+ + c^+ + t^+ + d^+ + s^+ + b^+ \\ D_{\Delta\Sigma} \equiv u^+ + c^+ + t^+ - d^+ - s^+ - b^+ \end{pmatrix}$
  - Non-Singlet:

$$q_i^{\text{NS}} = \left\{D_{uc}, D_{ds}, D_{sb}, D_{ct}, u^-, d^-, s^-, c^-, b^-, t^-\right\}, i = 1, \dots, 10$$

where $D_{ij} = q_i^+ - q_j^+$ and $q^\pm \equiv q \pm \bar{q}$.

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where $D_{ij} = q_i^+ - q_j^+$ and $q^\pm \equiv q \pm \bar{q}$.

- The Singlet sector:

$$\nu^2 \frac{\partial}{\partial \nu^2} \begin{pmatrix} \gamma \\ \Sigma \\ D_{\Delta\Sigma} \end{pmatrix} = \frac{\alpha(\nu)}{4\pi} \begin{pmatrix} e_\Sigma^2 P_{\gamma\gamma}^{(0)} & \eta^+ P_{\gamma q}^{(0)} & \eta^- P_{\gamma q}^{(0)} \\ \theta^- P_{q\gamma}^{(0)} & \eta^+ P_{qq}^{(0)} & \eta^- P_{qq}^{(0)} \\ \theta^+ P_{q\gamma}^{(0)} & \eta^- P_{qq}^{(0)} & \eta^+ P_{qq}^{(0)} \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \Sigma \\ D_{\Delta\Sigma} \end{pmatrix}$$

- The Non-Singlet sector:

$$\nu^2 \frac{\partial}{\partial \nu^2} q_i^{\mathsf{NS}}(x, \nu) = e_i^2 P_{qq}^{(0)}(x) \otimes q_i^{\mathsf{NS}}(x, \nu)$$

Combining the QCD and QED evolution operators

- The solution of the QED DGLAP is then

$$\underbrace{\begin{pmatrix} g(\mu,\nu) \\ \mathbf{q}^{\text{SG}}(\mu,\nu) \\ q_1^{\text{NS}}(\mu,\nu) \\ \vdots \\ q_{10}^{\text{NS}}(\mu,\nu) \end{pmatrix}}_{\mathbf{q}(\mu,\nu)} = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{\Gamma}^{\text{SG}}_{\text{QED}} & 0 & \cdots & 0 \\ 0 & 0 & \Gamma^{\text{NS}}_{\text{QED},1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Gamma^{\text{N}}_{\text{QED},10} \end{pmatrix}}_{\mathbf{\Gamma}^{\text{QED}}(\nu,\nu_0)} \otimes \underbrace{\begin{pmatrix} g(\mu,\nu_0) \\ \mathbf{q}^{\text{SG}}(\mu,\nu_0) \\ q_1^{\text{NS}}(\mu,\nu_0) \\ \vdots \\ q_{10}^{\text{NS}}(\mu,\nu_0) \end{pmatrix}}_{\mathbf{q}(\mu,\nu_0)}$$

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- Finally in the modified QCD evolution basis $\tilde{q}_i = \{\gamma, g, \Sigma, V, V_3, V_8, \ldots\}$:
  - QED, with $\mathbf{q} = \mathbf{T}\cdot\tilde{\mathbf{q}}$:

$$\begin{aligned} \tilde{\mathbf{q}}(\mu,\nu) &= \left[\mathbf{T}^{-1}\cdot\mathbf{\Gamma}^{\text{QED}}(\nu,\nu_0)\cdot\mathbf{T}\right]\otimes\tilde{\mathbf{q}}(\mu,\nu_0) \\ &= \tilde{\mathbf{\Gamma}}^{\text{QED}}(\nu,\nu_0)\otimes\tilde{\mathbf{q}}(\mu,\nu_0) \end{aligned}$$

  - QCD:

$$\tilde{\mathbf{q}}(\mu,\nu) = \tilde{\mathbf{\Gamma}}^{\text{QCD}}(\mu,\mu_0)\otimes\tilde{\mathbf{q}}(\mu_0,\nu)$$